CONFORMAL CAPACITY AND QUASICONFORMAL MAPPINGS IN $\bar{R}^n$*

BY
URI SREBRO

ABSTRACT

Conformal capacity of some ring domains in n-spaces is computed and applied in estimations of the modulus of Groetzsch and Teichmuller's domains. Upper bounds for mod R are given here by means of the specific diameters of the ring domain R. Convergence theorems of Lehto and Virtanen are generalized here to n-spaces $n > 2$; and are applied to discrete groups of n-Moebius transformations.

0. Introduction

We consider here the n-Moebius space $\bar{R}^n$, i.e. the one point compactification of R^n, and quasiconformal homeomorphisms in $\bar{R}^n$. Continuity and convergence of mappings in $\bar{R}^n$ are regarded here with respect to the spherical metric in $\bar{R}^n$. An arc element at a point x, $x \in \bar{R}^n$, with respect to the spherical metric, is given by $ds = dx(1 + |x|^2)^{-1}$. The metric is compatible with the topology of $\bar{R}^n$ and it is conformal. Hence a mapping is K-quasiconformal in the spherical metric iff it is K-quasiconformal in the Euclidean metric. $\bar{R}^n$ with the spherical metric is isometric to an n-sphere S_n in R^{n+1}, hence properties of $\bar{R}^n$ or of mappings of $\bar{R}^n$, which are expressed by means of the spherical metric have simple interpretations in S_n.

After some preliminaries in spherical geometry we present two examples of the calculation of conformal capacity. These examples are then applied in establishing new estimates to the conformal modulus of the extremal ring domains of Groetzsch

---

* This paper is mainly based on the author's Ph.D. Thesis done at the Technion, Haifa, under the guidance of Professor E. Netanyahu. The author wishes to thank Professor E. Netanyahu for his help in preparing this paper.

Received September 12, 1969 and in revised form May 26, 1970

and Teichmüller. Lehto and Virtanen present in [6] estimates of the conformal modulus of a ring domain R in $\bar{R}^2$ by means of the spherical diameters of the boundaries of R and the spherical distance between these boundaries. The method by which these estimates are obtained in [6] cannot be carried on to ring domain R in $\bar{R}^n$, $n > 2$, unless R is bounded by spheres. These estimates are generalized here for $\bar{R}^n$, $n > 2$, by other methods.

With the aid of these estimates two theorems of [6], about the convergence of K-quasiconformal homeomorphisms in $\bar{R}^2$, are generalized to the following:

THEOREM 3. *Let F be a family of K-quasiconformal homemorphisms of a domain $D \subset \bar{R}^n$, $n \geqq 2$, into $\bar{R}^n$. If each $f \in F$ omits in D two points a_f, and b_f of spherical distance greater than some $d > 0$, then F is normal and equicontinuous in D.*

THEOREM 4. *If $\{f_m\}$ is a sequence of K-quasiconformal homeomorphisms of a domain $D \subset \bar{R}^n$ into $\bar{R}^n$, which converges to a limit function f, then one of the following cases occurs:*

a) *f is K-quasiconformal homeomorphism of D. The convergence is uniform on every compact subset of D.*

b) *$f[D]$ consists of two points x and y. f maps one point $b \in D$ onto y and maps $D - \{b\}$ onto x. The convergence is uniform on every compact subset of $D - \{b\}$.*

c) *$f[D]$ consists of one point x.*

It is added in [6] that if case (c) of Theorem 4 holds in $\bar{R}^2$, then uniform convergence is not guaranteed. We prove here, however, that in this case, i.e. when $f_m(z) \to x$ for every $z \in \bar{R}^n$, $\{f_m\}$ has a subsequence, which converges uniformly in any compact subset of $\bar{R}^n - \{y\}$ where y is some excluded point, possibly $x = y$. We then study the convergence of $\{f_m^{-1}\}$, where $\{f_m\}$ is a convergent sequence of K-quasiconformal automorphisms of $\bar{R}^n$.

As an application to the convergence theorems we prove a theorem about groups of Moebius transformations in $\bar{R}^n$.*

1. The spherical metric in $\bar{R}^n$

1.1 The length of an arc element in the spherical metric at finite points is defined by

---

* Theorems 3, 4 and 6 here, which generalize results of [6] were obtained independently by Väisälä [8] with quite the same proofs.

$$ds = g(x)|dx|,$$

where

$$g(x) = (1 + |x|^2)^{-1}.$$

Here $|x|^2 = \Sigma_{i=1}^{n} x^2$, where x_i, $i = 1, \cdots, n$, are the coordinates of x, with respect to an orthonormal base in R^n.

At $x = \infty$ $ds = dy$ where $y = f(x)$ is the inversion with respect to $|x| = 1$.

Let C be an arc, Σ an $(n-1)$ hypersurface and D a domain in $\bar{R}^n$. The corresponding spheric measures of these sets are

$$L(C) = \int_C g|dx|, \; A(\Sigma) = \int_\Sigma g^{n-1} d\sigma, \; V(D) = \int_D g^n dV,$$

and will be called spheric length, area and volume respectively. Here $d\sigma$ and dV denote the $(n-1)$ and n dimensional elements respectively.

The spherical distance between two points a and b is

$$k(a,b) = \inf_C L(C),$$

where the inf is taken over all arcs C in $\bar{R}^n$, which join a and b.

1.2 The stereographic projection f of $\bar{R}^n$ onto the sphere

$$S_n = \{u : |u|^2 = u_{n+1}\} \subset R^{n+1}$$

is given by

$$u_i = x_i(1 + |x|^2)^{-1}, \; i = 1, \cdots, n;$$
$$u_{n+1} = |x|^2(1 + |x|^2)^{-1}$$
$$f(\infty) = (0, \cdots, 0, 1).$$

f is conformal and maps hyperspheres and hyperplanes of $\bar{R}^n$ onto hyperspheres on S_n. Furthermore, the spherical length of an arc element in $\bar{R}^n$ is equal to the length of the corresponding element on S_n. Hence, $\bar{R}^n$ with the spherical structure is isometric to S_n.

1.3 By a sphere in $\bar{R}^n$ we shall mean a $(n-1)$-sphere or a $(n-1)$-plane. By a *great subsphere* on a given sphere S is meant a $(n-2)$-sphere which is common to S and an $(n-1)$ plane which passes through the center of S.

The characterization of the automorphisms of $\bar{R}^n$ which preserve the spherical measure is presented in the following lemma:

LEMMA 1. a) *Inversion with respect to a sphere, which cuts the unit sphere $|x| = 1$ in a great sub-sphere, preserves the spherical measure.*

b) *Any automorphism f of $\bar{R}^n$ which preserves the spherical measure is necessarily a finite composition of inversions with respect to spheres, which cut the unit sphere in great sub-spheres.*

PROOF. a) Let A be a sphere in $\bar{R}^n$ which cuts the unit sphere $|z| = 1$, in a great sub-sphere. The proof is trivial when A is a hyper-plane, so let a be the center of A and R be the radius of A; then

$$R^2 = 1 + |a|^2. \tag{1}$$

Let $y = f(x)$ denote the inversion with respect to A. Since f is conformal the limit

$$\lim_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|}$$

exists and is independent of how $h \to 0$. We call this limit the derivative of f and denote it by $f'(x)$. To establish (a), one must prove that

$$(1 + |x|^2)^{-1} = f'(x)\,(1 + |f(x)|^2)^{-1}. \tag{2}$$

Equation (2) is easily obtained by considering a 2-hyperplane π which passes through the points 0, a, x and $y = f(x)$. The restriction of f to π is then given by:

$$f|\pi = a + R^2(\bar{a} - \bar{x}_2)^{-1}. \tag{3}$$

Here, a, x are regarded as complex numbers. f is conformal and $f|\pi$ is isogonal, hence

$$f'(x) = \left|\frac{\partial f|\pi}{\partial x}\right| = R^2|x - a|^{-2}. \tag{4}$$

One obtains the required relation (2) by applying (1) (3) and (4).

b) Since the spherical metric is conformal, and f preserves the spherical measure, f must be conformal and by a theorem of Liouville it must be a Moebius transformation. If $f(\infty) = \infty$, then f maps every hyperplane which passes through 0 onto a hyperplane of the same kind, f maps therefore every sphere with a center at 0 onto itself. Consequently, f can be represented by reflections with respect to a finite number of $(n-1)$-hyperplanes each of which passes through 0, and thus cuts the unit sphere $|x| = 1$ in a great sub-sphere.

If $f(\infty) = a \neq \infty$ we consider a sphere with the center at a which cuts $|x| = 1$ in a great sub-sphere. Let g denote the inversion with respecṭ to this sphere.

$g(f(\infty)) = \infty$ and $g \circ f$ preserves the spherical measure. $g \circ f$ can therefore be represented by inversions with respect to a finite number of $(n-1)$-hyperplanes which pass through 0. f is thus obtained by another inversion with respect to a sphere which cuts $|x| = 1$ in a great sub-sphere. This completes the proof of the Lemma.

DEFINITION. By spherical motion we denote a finite composition of inversions with respect to spheres which cut the unit sphere $|x| = 1$ in great sub-spheres.

The last lemma merely says that the group of spherical isometrics consists of all spherical motions.

1.4 The spherical geodesics are characterized in the following lemma

LEMMA 2. *The geodesics in $\bar{R}^n$ with respect to the spherical metric are circles which cut the unit sphere in antipodal points.*

PROOF. Let a and b be two points in $\bar{R}^n$. We prove that

$$k(a,b) = \inf_c \int_c |dx|(1+x^2)^{-1} = \int_{c_0} |dx|.(1+x^2)^{-1}$$

where c is an arc which joins the points a and b, and c_0 is part of a circle which passes through a and b and meets the unit sphere in antipodal points. This is trivial if $a = \infty$ or $b = \infty$. If $a, b \neq \infty$ we map a to ∞ by a spherical motion. It is then adequate to show that an inversion f with respect to a sphere A, which cuts the unit sphere in a great subsphere maps any line l which passes through 0 onto a circle which meets $|x| = 1$ in antipodal points. Let π be a 2-hyperplane which passes through l. It is then elementary to verify that the restriction of f to π, which is an inversion with respect to $\pi \cap A$ maps l onto a circle which cuts the unit circle of π in antipodal points, and these points are also antipodal on the unit sphere $|x| = 1$.

2. The conformal capacity and the modulus of domains in $\bar{R}^n$

2.1 Let D be a domain in $\bar{R}^n$ and let B_0 and B_1 be two compact disjoint sets in D. The conformal capacity of D with respect to B_1 and B_2 is defined (cf. [9]) by

$$\Gamma(D; B_0, B_1) = \inf_u \int_D |\nabla u|^n d\tau,$$

where the inf is taken over all real functions u which are continuously differentiable in D with the boundary values 0 on B_0 and 1 on B_1. Following Loewner [7], the conformal capacity of a ring domain R in $\bar{R}^n$, i.e. a domain whose complement CR consists of two connected components c_0 and c_1, is defined by

$$\Gamma(R) = \Gamma(R; \partial c_0, \partial c_1).$$

The modulus of R (cf. [2]) is defined by

$$\operatorname{mod} R = \left(\frac{\omega_n}{\Gamma(R)}\right)^{1/(n-1)}.$$

According to these definitions the modulus of the spherical ring domain $a < |x| < b$ is $\log b/a$, which agrees with the classical definition of the modulus of ring domains in the plane. Furthermore mod R is conformally invariant. We give two examples of the calculation of the conformal capacity.

EXAMPLE 1. *Let G be the cylinder $\sum_{i=2}^n x_i^2 < R^2$, $0 < x_1 < h$. We denote its axis of symmetry : $0 \leqq x_1 \leqq h$, $x_i = 0$ $(i = 2,\cdots,n)$, by L; its base $\sum_{i=2}^n x_i^2 \leqq R^2$, $x_1 = 0$ by B; its envelope $\sum_{i=2}^n x_i^2 = R$, $0 \leqq x_1 \leqq h$, by M. The conformal capacity of G with respect to L and M is given by*

$$\Gamma(G; L, M) = \frac{h}{R} \cdot \frac{\omega_{n-1}}{(n-1)^{n-1}}$$

PROOF. We use cylindrical coordinates $(r, \theta_1, \cdots, \theta_{n-2}, z)$ such that G is defined by $r < R$, $0 < z < h$; L is defined by $r = 0, 0 \leqq z \leqq h$; and the envelope M is defined by $r = R$, $0 \leqq z \leqq h$.

Let u be a continuously differentiable function in G, which has the boundary values 0 on L and 1 on M. By integrating along a line orthogonal to L and applying Hölder's inequality we have:

$$1 \leqq \int_0^R |\nabla u| dr \leqq \left(\int_0^R r^{n-2} |\nabla u|^n dr\right)^{1/n} \left(\int_0^R r^{(2-n)/(n-1)} dr\right)^{(n-1)/n},$$

Hence,

$$\int_0^R r^{n-2} |\nabla u|^n dr \geqq \left(\int_0^R r^{(2-n)/(n-1)} dr\right)^{1-n} = R^{-1}(n-1)^{1-n}.$$

Therefore,

$$\int_G |\nabla u|^n dv = \int d\theta \int_0^h dz \int_0^R r^{n-2} |\nabla u|^n dr \geqq \frac{h}{R} \cdot \frac{\omega_{n-1}}{(n-1)^{n-1}}. \tag{1}$$

Here $d\theta$ is the differential which belongs to $\theta_1, \cdots, \theta_{n-2}$.

On the other hand the function $u(x) = (r/R)^{1/(n-1)}$ is admissable and gives equality in (1). This completes the proof.

EXAMPLE 2. *Let R be the half spherical ring $a < |x| < b$, $x_1 > 0$. We denote its axis of symmetry $a \leqq x_1 \leqq b$, $x_i = 0 (i = 2, \cdots, n)$, by L; its base $a \leqq |x| \leqq b$, $x_1 = 0$ by B. Then the conformal capacity of R with respect to L and B is given by*

$$\Gamma(R; L, B) = C \log b/a,$$

where,

$$C = \omega_{n-1} \left[\int_0^{\pi/2} (\sin t)^{(2-n)/(n-1)} dt \right]^{1-n}.$$

PROOF. We use spherical coordinates $(r, \theta, \theta_2, \cdots, \theta_{n-1})$ where θ denotes the angle between the x_1 axis and the vector x. The element dv is given by

$$dv = r^{n-1} (\sin\theta)^{n-2} dr d\theta d\phi,$$

where $d\phi$ is the differential which belongs to the last $(n-2)$ angles $\theta_2, \cdots, \theta_{n-1}$ The half ring R is defined by $a < r < b$, $0 \leqq \theta < \pi/2$; B is defined by $a \leqq |r| \leqq b$, $\theta = \pi/2$; L is defined by $a \leqq |r| \leqq b$, $\theta = 0$. Let u be a continuously differentiable function in R which has the boundary values 0 on L and 1 on B. Integrating from L to B along a circular arc of radius r, on which θ varies from 0 to $\pi/2$ while all other coordinates remain fixed, and applying Hölder's inequality we have:

$$\frac{1}{r} \leqq \int_0^{\pi/2} |\nabla u| d\theta \leqq \left(\int_0^{\pi/2} (\sin\theta)^{n-2} |\nabla u|^n d\theta \right)^{1/n} \left(\int_0^{\pi/2} (\sin\theta)^{(2-n)/(n-1)} d\theta \right)^{(n-1)/n}.$$

Hence

$$\begin{aligned} (2) \quad \int_R |\nabla u|^n dv &= \int d\phi \int_a^b r^{n-1} dr \int_0^{\pi/2} (\sin\theta)^{n-2} |\nabla u|^n d\theta \\ &\geqq \int d\phi \int_a^b \frac{dr}{r} \left[\int_0^{\pi/2} (\sin\theta)^{(2-n)/(n-1)} d\theta \right]^{1-n} = C \log \frac{b}{a}, \end{aligned}$$

where

$$C = \omega_{n-1} \left[\int_0^{\pi/2} (\sin\theta)^{(2-n)/(n-1)} d\theta \right]^{1-n}.$$

We note that the last integral exists for every $n \geqq 2$. On the other hand the function

$$u(x) = \frac{\int_0^{\theta} (\sin t)^{(2-n)/(n-1)} dt}{\int_0^{\pi/2} (\sin t)^{(2-n)/(n-1)} dt}$$

is admissable in R and gives equality in (2). This completes the proof.

2.2 The examples of 2.1 are now applied to establish new estimates for the moduli of the extremal ring domains of Groetzsch and Teichmüller. *Groetzsch ring* domain $R_G(a)$, $a < 1$, is the domain bounded by the unit sphere $|x| = 1$ and the segments $0 \leqq x_1 \leqq a$, $x_i = 0$ $(i = 2, \cdots, n)$.

Teichmuller ring domain $R_T(b)$, $b > 0$ is the complement of the two colinear segments $-1 \leqq x_1 \leqq 0$, $x_i = 0$ $(i = 2, \cdots, n)$ and $b \leqq x_1 \leqq \infty$, $x_i = 0$ $(i = 2, \cdots, n)$.

First estimate for mod $R_G(a)$: $\operatorname{mod} R_G(a) \leqq A(1/a)^{1/(n-1)}$, *where*

$$A = (n-1)\left(\frac{\omega_n}{\omega_{n-1}}\right)^{1/(n-1)}$$

PROOF. Let R_1 be the half ball $|x| < 1, x_1 < 0$; R_2 be the cylinder $|x|^2 < x_1^2 + 1$, $0 < x_1 < a$; and R_3 be the half ball $|x - (a, 0, \cdots, 0)| < 1$, $x_1 > a$. Denote the segment $0 \leqq x_1 \leqq a$, $x_i = 0$ $(i = 2, \cdots, n)$ by L. The ring domain $R = R_1 \cup R_2 \cup R_3 - L$ contains $R_G(a)$; hence, by the monotonicity of the modulus (see [2, p. 914]): $\operatorname{mod} R_G(a) \leqq \operatorname{mod} R$. Applying example 1, we have

$$\Gamma(R) = \inf_u \int_R |\nabla u|^n dv \geqq \inf_u \int_{R_2} |\nabla u|^n dv = \Gamma(R_2; L, M) = a\omega_{n-1}(n-1)^{1-n},$$

where the inf on the left hand of the inequality is taken over all function u admissable in R, while on the right hand the inf is taken over all functions u which are admissable in the cylinder R_2. Here M is the envelope of R_2. The required estimate is an immediate consequence of the last inequality.

Second estimate for mod $R_G(a)$: $\operatorname{mod} R_G(a) \leqq \gamma(\log(1+a)/(1-a))^{1/(1-n)}$ *where*

$$\gamma = \frac{\left(\dfrac{\omega_n}{\omega_{n-1}}\right)^{1/(n-1)}}{\displaystyle\int_0^{\pi/2} (\sin t)^{(2-n)/(n-1)} dt}.$$

(cf. [1, p. 20] *where the same estimate is obtained for* $n = 3$).

PROOF. Inversion with respect to a sphere with center at the point $(-1, 0, \cdots, 0)$ and radius $= 2$ maps $R_G(a)$ on the half space $x_1 > 1$ minus the segment $(3-a)/(1+a) \leqq x_1 \leqq 3$ $x_i = 0$ $(i = 2, \cdots, n)$. By further translation and stretching, which are conformal and thus preserve the modulus, the last ring domain can be mapped on the ring domain R which is bounded by $x_1 = 0$ and the segment

$1-a \leqq x_1 \leqq 1+a$, $x_i = 0$ $(i=2,\cdots,n)$. Hence, mod $R_G(a) =$ mod R. Let R_1 be the half ball $|x| < 1-a$, $x_1 > 0$; R_2 be the half annulus $1-a < |x| < 1+a$, $x_1 > 0$; and R_3 the domain $|x| > 1+a$, $x_1 > 0$. Denote by L the segment $1-a \leqq x_1 \leqq 1+a$, $x_i = 0$ $(i=2,\cdots,n)$ and by B the base of R_2, i.e. the set $1-a \leqq |x| \leqq 1+a$, $x_1 = 0$. Applying example 2 we have

$$\Gamma(R) = \inf_u \int_R |\nabla u|^n dv \geqq \inf_u \int_{R_2} |\nabla u|^n dv = \Gamma(R_2; L, B)$$

$$= \omega_{n-1}\left[\int_0^{\pi/2} (\sin t)^{(2-n)/(n-1)} dt\right]^{1-n} \log\frac{1+a}{1-a},$$

where the inf on the left hand of the inequality is taken over all admissable functions u in R, while the inf on the left hand is taken over all admissable functions in R_2. The required estimate is readily obtained from the last inequality.

In view of the relation mod $R_T(a) = 2\,\mathrm{mod}\, R_G(1+a)^{1/2}$ the following estimates are obtained:

First estimate for mod $R_T(a)$: mod $R_T(a) \leqq 2A(1+a)^{1/2n-2}$,

Second estimate for mod $R_T(a)$:

$$\mathrm{mod}\, R_T(a) \leqq 2\gamma\left(\log\frac{(1+a)^{1/2}+1}{(1+a)^{1/2}-1}\right)^{1/(1-n)}$$

A and γ have the same meaning as above.

As a consequence of the second estimate for mod $R_T(a)$ one has the following

COROLLARY. $\lim_{a\to 0} \mathrm{mod}\, R_T(a) = 0.$

Teichmuler's ring domain is extremal in the following sense: Let R be a ring domain in $\bar{R}^n$ and let C_0 and C_1 be the components of the complement of R. If C_0 has two points p and p' of distance a, and C_1 contains ∞ and has a point q within a distance b from p, then mod $R \leqq \mathrm{mod}\, R_T(b/a)$ (see [2], [3]).

3.2 The following two theorems generalize estimates which are proved in [6] for ring domains in $\bar{R}^2$.

THEOREM 1. *Let R be a ring domain in $\bar{R}^n$, and let C_0 and C_1 be the connected components of CR. If the spheric diameters of C_0 and C_1 are greater than a positive member d, then*

$$\mathrm{mod}\, R \leqq 4A\left(\frac{1}{d}\right)^{1/(n-1)}$$

where

$$A = (n-1)\left(\frac{\omega_n}{\omega_{n-1}}\right)^{1/(n-1)}.$$

PROOF. Since C_0 and C_1 are connected and of spheric diameters greater than d, there are points $p,\ p' \in C_0$ and $q, q' \in C_1$ such that $k(p,p') = k(q,q') = d$. Since there is a spheric motion which maps q' onto ∞ and obviously preserves the spheric measure and the conformal modulus, we shall assume $q' = \infty$. We also may assume $|p| < |p'|$, otherwise change the roles of p and p'.

$$d = k(q,\infty) = \int_q^\infty (1+|x|^2)^{-1}|dx| = \frac{\pi}{2} - \operatorname{arctg}|q|.$$

Hence $|q| = ctg\, d \leqq d^{-1}$, and therefore

$$|p-q| \leqq |p| + d^{-1}.$$

$$d = k(p,p') = \inf_C \int_C (1+|x|^2)^{-1}|dx| \leqq |p-p'|\ (1+|p|^2)^{-1}.$$

Noting that $d \leqq \pi/2$ we have:

$$\frac{|p-q|}{|p-p'|} \leqq \frac{d^{-1}+|p|}{(1+|p|^2)d} < \frac{3}{2d^2}.$$

From the extremal property of Teichmuller's ring domain R_T and the monotonicity of mod $R_T(a)$ as a function of a we have by using the first estimate for $\operatorname{mod} R_T(a)$ and the last inequality:

$$\operatorname{mod} R \leqq \operatorname{mod} R_T\left(\frac{|p-q|}{|p-p'|}\right) \leqq \operatorname{mod} R_T\left(\frac{3}{2d^2}\right) < 2A\left(1+\frac{3}{2d^2}\right)^{1/(2n-2)},$$

where

$$A = (n-1)\left(\frac{\omega_n}{\omega_{n-1}}\right)^{1/(n-1)}.$$

But

$$1 + \frac{3}{2d^2} \leqq \frac{\pi^2}{4d^2} + \frac{3}{2d^2} < \frac{4}{d^2},$$

hence

$$\operatorname{mod} R < 4A\left(\frac{1}{d}\right)^{1/(n-1)}.$$

THEOREM 2. *Let R be a ring domain in $\bar{R}^n$. Let C_0 and C_1 be the connected components of the complement of R. If the spheric diameters of C_0 and C_1 are greater than $2\rho, \rho > 0$, and the spherical distance between C_0 and C_1 is e, $0 < e < \rho/2$, then*

$$\operatorname{mod} R \leqq 2\gamma\left(\log\frac{\rho^{3/2}}{4e}\right)^{1/(1-n)},$$

where γ is the constant given above in the estimates for $\operatorname{mod} R_G(a)$.

PROOF. Since C_0 and C_1 are compact and of spherical distance e, there are two points p in C_0 and q in C_1 such that $k(p,q) = e$. C_0 and C_1 are connected and of spherical diameters less than d, there are, therefore, points p' in C_0 and q' in C_1 such that $k(p,q') = k(q,q') = \rho$. Since q' can be mapped onto ∞ by means of a spherical motion which does not change the spherical measure and the modulus, we may assume that $q' = \infty$.

Denote $|p - p'| = a$, $|p - q| = b$.
We have

$$\rho = k(p,p') \leqq |p - p'| = a. \tag{1}$$

On the other hand

$$\rho = k(q,\infty) = \int_{|q|}^{\infty}(1 + r^2)^{-1}dr = \frac{\pi}{2} - \operatorname{arctg}|q|.$$

Hence

$$|q| = \operatorname{ctg}\rho.$$

$$e = k(p,q) = \inf_c \int_c (1 + |x|^2)^{-1}|dx| \geqq \int_{|q|}^{|q|+b}(1 + r^2)^{-1}dr = \operatorname{arctg}(b + \operatorname{ctg}\rho) - \operatorname{arctg}(\operatorname{ctg}\rho).$$

The inf is taken here over all arcs c which join p and q; the integral on the left hand of the inequality is taken over a radial segment of length $b = |p - q|$ which starts at the point q.
Thus we have

$$\operatorname{tg} e \geqq b[1 + (b + \operatorname{ctg}\rho)\operatorname{ctg}\rho]^{-1},$$

or

$$b(1 - \operatorname{tg} e \operatorname{ctg}\rho) \leqq \operatorname{tg} e(1 + \operatorname{ctg}^2\rho).$$

Given $e < d/4$ we have $1 - \text{tg}e \ \text{ctg}\rho > 0$, and therefore, with the aid of the last inequality, and (1)

$$\frac{b}{a} \leqq \frac{\sin e}{\sin \rho \sin(\rho - e)} \leqq \frac{2e}{\rho \sin(\rho - e)}, \tag{2}$$

From the extremal property of Teichmuller's ring domain we have

$$\operatorname{mod} R \leqq \operatorname{mod} R_T\left(\frac{b}{a}\right),$$

and by applying (2) and the second estimate for mod $R_T(a)$ the required estimate is readily obtained.

COROLLARY. *With the assumptions of Theorem* 2 mod $R \to 0$ *as* $e \to 0$ *while* $\rho \geqq \rho_0 > 0$.

3. Convergence of K-quasiconformal homeomorphisms in $\bar{R}^n$

3.1 *K-quasiconformality* is here in the sense of Gehring [4] according to which a homeormorphism f of a domain $D \subset \bar{R}^n$ into $\bar{R}^n$ is called K-quasiconformal iff

$$\sup_{RD} \left(\frac{\operatorname{mod} f[R]}{\operatorname{mod} R}, \frac{\operatorname{mod} R}{\operatorname{mod} f[R]}\right) = K < \infty$$

where the sup is taken over all ring domains R in D. *Continuity* and *convergence* will be always with respect to the *spherical metric.*

Let F be a family of homeomorphisms f of a domain $D \subset \bar{R}^n$. F is said to be *equicontinuous* at a point $x_0 \in D$, iff for every $\varepsilon > 0$ there is a neighborhood U of X_0, such that $k(f(x), f(x_0)) < \varepsilon$ for every $x \in U$ and every $f \in F$. F is equicontinuous in D, iff it is equicontinuous at each $x \in D$. F is said to be *normal* in D, iff every infinite sequence of elements of F has a subsequence which converges uniformly in every compact subset of D.

Since $\bar{R}^n$ is compact and separable, equicontinuity implies normality. The converse is also true, which can be easily proved. Thus equicontinuity and normality are equivalent in $\bar{R}^n$.

3.2. The following two theorems (3 & 4) are generalizations of results which are presented in [6 Ch. 11 § 5] for homeomorphisms in $\bar{R}^2$. The proof of Theorem 3 (below) is based on the estimates which have been derived here in 2.3. The argument in the proofs of both theorems are essentially due to [6]. These two theorems appear also in [8].

THEOREM 3. *Let F be family of K quasiconformal homeomorphisms of a domain $D \subset \bar{R}^n$ into $\bar{R}^n$. If each $f \in F$ omits two points a_f and b_f of spherical distance greater than a positive number d, then F is equicontinuous and normal in D.*

PROOF. Let $x_0 \in D$ and $0 < \varepsilon < d$. Let $0 < r_1 < r_2$ be such that $U = \{x: k(x, x_0) < r_2\} \subset D$ and the ring domain $R = \{x: r_1 < k(x, x_0) < r_2\}$ is such that

$$\text{mod}\, R > 4Ak\, \varepsilon^{1/(1-n)}, \tag{1}$$

where A is the constant which is defined in Theorem 1. f is K-quasinconformal hence

$$\text{mod}\ R < K \,\text{mod}\, f[R]. \tag{2}$$

Now let x be any point in U. Denote $d' = k(f(x), f(x_0))$. $f[R]$ is a ring domain which separate $f(x)$ and $f(x_0)$, from two points a_f and b_f of spherical distance greater then d, therefore the complement of $f[R]$ has two components of spherical diameters $\geqq \min(d, d')$. Hence, by Theorem 1, we have

$$\text{mod} f[R] \leqq 4A\,[\min(d, d')]^{1/(1-n)}. \tag{3}$$

From (1), (2) and (3) we have $\min(d, d') < \varepsilon$. But, $\varepsilon < d$, therefore $d' < \varepsilon$. We have proved that F is equicontinuous in D, hence F is also normal in D.

3.3 THEOREM 4. *If $\{f_m\}$ is a sequence of K-quasiconformal homeomorphisms of a domain $D \subset \bar{R}^n$ into $\bar{R}^n$, which converges to a limit function f, then one of the following cases occurs:*

a) *f is a K-quasiconformal homeomorphism of D. The convergence is uniform on every compact subset of D.*

b) *$f[D]$ consists of two points x and y. f maps one point $b \in D$ onto y and maps $D - \{b\}$ onto x. The convergence is uniform on every compact subset of $D - \{b\}$.*

c) *$f[D]$ consists of one point x.*

PROOF. If $f[D]$ consists of two points x and y, then D has points a and b such that $f_m(a) \to x$ and $f_m(b) \to y$. Therefore $k(f_m(a), f_m(b))$ is bounded away from 0, and in view of Theorem 3, $\{f_m\}$ is equicontinuous in $D - \{a\} - \{b\}$. Consequently $\{f_m\}$ converges uniformly on every compact subset of $D - \{a\} - \{b\}$. It follows that either $f[D - \{a\} - \{b\}] = x$ or $f[D - \{a\} - \{b\}] = y$. We may assume $f[D - \{a\} - \{b\}] = x$. By the same argument, $\{f_m\}$ is equicontinuous in

$D - \{a'\} - \{b\}$, where a' is a point of $D - \{a\} - \{b\}$. And consequently, $\{f_n\}$ converges uniformly in every compact subset of $D - \{b\}$, as stated in (b).

If $f[D]$ has more than two points, then D has three distinct points a_i, $i = 1,2,3$ such that for every m and $i \neq j$ $k(f_m(a_i), f_m(a_j)) > d$ for some positive number d. If follows, by Theorem 3, that $\{f_m\}$ is equicontinuous in D, and therefore it converges uniformly on every compact subset of D. Hence f is continuous. To prove that f is topological it will be adequate to show that f is (1-1). Suppose $f(a) = f(b)$ for some distinct points $a, b \in D$. Let S be a sphere contained in D and separates a and b. $\{f_m\}$ converges uniformly on S. Since $f[S]$ separates $f_m(a)$ and $f_m(b)$, it follows that S has at least one point which is mapped by f onto $f(a) = f(b)$. Hencc every neighborhood U of a has at least one point other than a which is mapped by f onto $f(a)$. We now show that f is either (1-1) in U or it is constant in U. Indeed, $\{f_m\}$ is equicontinuous in D, therefore, if U is sufficiently small, then the spherical diameter of $f[U] < \pi/4$. Suppose U has distinct points p, p' and q, such that $f(q) = f(p) = f(p')$. Join q with the boundary of U and p with p' by means of disjoint arcs γ_1, $\gamma_2 \subset U$. Denote by R the ring domain $U - (\gamma_1 \cup \gamma_2)$. For every m sufficiently large the complements of $f_m[R]$ has two components of spherical diameters which are bounded away from zero while the spherical distance between them tends to 0, hence by corollary at the end of 2.3 $\mathrm{mod} f_m(R) \to 0$, which contradicts $\mathrm{mod} f_m(R) \geqq K^{-1} \mathrm{mod} R > 0$. We have proved that the restriction of f to U is either (1-1) or a constant. It follows that the set of points where f is (1-1) is open and the set of points where $f \equiv$ const. is open. Since D is connected, then f is either (1-1) in D or $f \equiv$ const. in D. $f \not\equiv$ const in D, therefore f is (1-1) in D, and therefore f is a homeomorphism. The K-quasiconformality of f follows from the continuity of the modulus of ring domains and the uniform convergence of $\{f_m\}$ (see [5, lemma 6] and [2]). We proved that if $f[D]$ has more than two points, then (a) must occur. This completes the proof.

Noting that spherical and Euclidean metrics are equivalent in the finite space R^n, it is not hard to see that Theorems 3 and 4, here, generalize some of the results of [5] and [2]. Theorem 4 here, is more general since it includes the case $D = \bar{R}^n$.

3.4 In case (c) of Theorem 4 i.e. when $f_m(z) \to x$ for every $z \in D$, uniform convergence is not guaranted in any subset of D. Moreover, we show now by an example, that for any given finite set of points $\{a_i\}$ $i = 1, \cdots, k$ in $\bar{R}^n$ we can construct a sequence $\{f_m\}$ of K-quasiconformal automorphisms of $\bar{R}^n$ such that $f_m(z) \to \infty$ for every $z \in R^{-n}$, but the convergence is not uniform in any neighborhood of any of the points a_i.

For every $a_i \neq \infty$ $i = 1, \cdots, k$, define

$$f_{im}(z) = a_i + m^2(z - a_i) + (m - 1/m)a \qquad m = 1, 2, \cdots$$

where a is some fixed point of $\bar{R}^n$, $|a| = 1$. If $a_i = \infty$ then f_{im} is defined by $f_{im}(z) = z + m$. Evidently for every fixed $i, f_{im}(z) \to \infty$ as $m \to \infty$ for every $z \in \bar{R}^n$. The sequence $\{z + m: m = 1, 2, \cdots\}$ does not converge uniformly in any neighborhood of ∞; and for $a_i \neq \infty$ $f_{im}(z)$ has a fixed point at $z = a_i + am^{-1}$. Hence the sequence $\{f_{im}: m = 1, 2, \cdots\}$ does not converge uniformly in any neighborhood of a_i. Now arrange $\{f_{im}\}, i = 1, \cdots, k$, $m = 1, 2, \cdots$ in a sequence. This sequence converges to ∞ at any point of $\bar{R}^n$ and does not converge uniformly in any neighborhood of any of the points a_i.

However, if $D = \bar{R}^n$ in case (c) of Theorem 4, we prove the following Theorem.

3.5. THEOREM 5. *Let $\{f_m\}$ be a sequence of K-quasiconformal automorphisms. If $f_m(z) \to x$ for every $z \in \bar{R}^n$, then $\{f_m\}$ has a subsequence which converges uniformly in any compact subset of $\bar{R}^n - \{y\}$, where y is some excluded point, which depends on the subsequence, possibly $y = x$.*

PROOF. Let K_1 be a closed proper subset of $\bar{R}^n$ in which $\{f_m\}$ is not equicontinuous. Such a set exists, for otherwise $\{f_m\}$ would converge uniformly on $\bar{R}^n$, which is impossible as $f_m \to$ const. Thus there is a subsequence $\{f_m^{(1)}\}$ such that the spherical diameters of the sets $f_m^{(1)}[K_1]$, $m = 1, 2, \cdots$, are greater than some $d > 0$. Hence, by Theorem 3, $\{f_m^{(1)}\}$ is equicontinuous in CK_1.

Continuing similarily and recursively we end with closed sets K_i and infinite subsequences $\{f_m^{(i)}\}$, $i = 1, 2, \cdots$, such that $K_i \subset K_{i-1}$; the spherical diameter of K_i tends to zero as $i \to \infty$; $\{f_m^{(i-1)}\}$ is not equicontinuous in K_i and $\{f_m^{(i)}\}$ is a subsequence of $\{f_m^{(i-1)}\}$ which is equicontinuous in CK_i.

$\{f_m^{(m)}: m = 1, 2 \cdots\}$ is an infinite subsequence of $\{f_m\}$ and it is equicontinuous in $\cup CK_i = \bar{R}^n - \{y\}$ where $y = \cap K_i$. Consequently $\{f_m^{(m)}\}$ converges uniformly on every compact subset of $\bar{R}^n - \{y\}$.

3.6 With relation to Theorems 4 and 5 we prove now.

THEOREM 6. *Let $\{f_m\}$ be a sequence of K-quasiconformal automorphisms of $\bar{R}^n$, which converges to the limit mapping f.* (cf. [8]).

a) *If f is a K-quasiconformal automorphism of $\bar{R}^n$, then $f_m^{-1} \to f^{-1}$ uniformly in every compact subset of $\bar{R}^n$.*

b) *If the range of f consists of two points x and y, such that $f(b) = y$ and $f[\bar{R}^n - \{b\}] = x$, for some $b \in \bar{R}^n$, then $f_m^{-1}(z) \to b$ uniformly in every compact subset of $\bar{R}^n - \{x\}$.*

c) *If the range of* f *consists of one point* x *and* $f_m \to f$ *uniformly in every compact subset of* $\bar{R}^n - \{a\}$, *for some* $a \in \bar{R}^n$, *then* $f_m^{-1}(z) \to a$ *uniformly in every compact subset of* $\bar{R}^n - \{x\}$.

PROOF. a) Let x be any point of $\bar{R}^n$ and denote $y = f(x)$, i.e. $f_m(x) \to y$. Suppose $f_m^{-1}(y) \not\to x$, then $\{f_m^{-1}\}$ has a subsequence $\{f_{m_k}^{-1}\}$ such that $f_{m_k}^{-1}(y) \to x' \neq x$. By Theorem 4(a) $\{f_m\}$ is equicontinuous in x', hence $f_{m_k}(x') \to y$, but this is impossible since f is (1-1). Therefore $f_m^{-1} \to f^{-1}$. The convergence is uniform by Theorem 4(a).

b) Let U be any neighborhood of b and let K be any compact subset of $\bar{R}^n - \{x\}$. CU is compact and $b \notin CU$, hence $f_m[CU] \cap K = \emptyset$ for every m sufficiently large; thus $f_m[U] \supset K$ and consequently $f_m^{-1}[K] \subset U$.

c) The proof of (b) holds here too when we merely replace the point b by a.

Case (a) of the last theorem can be stated more generally to include sequences of K-quasiconformal homeomorphisms of domains D which are proper subsets of $\bar{R}^n$. In view of the remark at the end of 3.3, one can obtain this completion very easily from [5, p. 11] and [2]. The application to Theorems 3–6 which we present in the next section concerns, however, only conformal automorphism of $\bar{R}^n$.

3.7. Let G be a group of Moebius transformations in G. G is said to be *discrete* iff no sequence of distinct element of G converges to the identity. A point $x \in \bar{R}^n$ is said to be a *limit point* with respect to G iff there exists a point $a \in \bar{R}^n$ and a sequence $\{T_m\}$ of distinct elements of G, such that $T_m(a) \to x$. Other points are called *ordinary*. The set of all limit point is denoted by $L = L(G)$.

THEOREM 7. *Let* G *be a group of Moebius transformations in* $\bar{R}^n$ *and* $x \in L(G)$.

i) Gx *clusters at* x.

ii) *If* G *is discrete, then* G *has a sequence* $\{T_m\}$ *of distinct elements such that* $T_m(z) \to x$ *uniformly in every compact subset of* $\bar{R}^n - \{y\}$, *where* y *is some limit point, possibly* $x = y$.

PROOF. (i) is trivial when G is not discrete. Suppose G is discrete. $x \in L$, hence G has a sequence $\{T_m\}$, such that $T_m(a) \to x$ for some $a \in \bar{R}^n$. If $\{T_m(x)\}$ clusters at x, then the proof is finished; thus assume that no subsequence of $\{T_m(x)\}$ clusters at x. Since $\bar{R}^n$ is compact there is a point $y \in \bar{R}^n$, $y \neq x$, and a subsequence of $\{T_m\}$ which we denote again by $\{T_m\}$, such that $T_m(x) \to y$. $\{T_m\}$ is normal in $\bar{R}^n - \{a\} - \{x\}$, in view of Theorem 3; therefore $\{T_m\}$ has a subsequence, which we denote again by $\{T_m\}$, which converges to a mapping T uniformly in any compact subset of $\bar{R}^n - \{a\} - \{x\}$. Since G is discrete, T is singular*. In view of Theorem 4, either $T[\bar{R}^n - \{x\}] = x$ or $T[R^{-n} - \{a\}] = y$. In any case $T(x) = y$. In first case let $\{T_{m_i}\}$ be a subsequence such that $T_i \circ T_{m_i} \neq T_j \circ T_{m_j}$ for every $i < j$. Let $U_i = T_i \circ T_{m_i}$. For $\varepsilon > 0$ and sufficiently small let $V_\varepsilon(x) = \{z : k(z,x) < \varepsilon\}$, $V_\varepsilon(y) = \{z : k(z,y) \leqq \varepsilon)$ such that $V_\varepsilon(x) \cap V_\varepsilon(y) = \emptyset$. $T_i(z) \to x$ uniformly in the compact set $V_\varepsilon(y)$, and $T_i(x) \to y$, hence for every i sufficiently large $U_i(x) \in V_\varepsilon(x)$ and consequently $U_i(x) \to x$. If the latter case holds, i.e. $T(a) = x$ and $T[\bar{R}^n - \{a\}] = y$ we have, in view of Theorem 6, $T_m^{-1}(x) \to a$. T_1 (the first element in the given sequence) is continuous at the point a. Also $T_m^{-1}(x) \to a$, therefore there is an element $T_m \in \{T_m\}$, such that

$$k(T_1 \circ T_{m_1}^{-1}(x);\, T_1(a)) < 1.$$

Denote $U_1 = T_1 \circ T_{m_1}^{-1}$. For natural k, let $U_k = T_k \circ T_{m_k}^{-1}$ such that $U_k \neq U_j$ for every $j < k$, and such that

$$k(U_k(x), T_k(a)) < 1/k.$$

Since $T_m(a) \to x$ it then follows that $U_k(x) \to x$, and the proof of (i) completed.

(ii) If $x \in L(G)$, then, by (i), G has a sequence $\{T_m\}$ such that $T_m(x) \to x$. If $T_m(z) \to x$ for every $z \in \bar{R}^n$ then, by Theorem 5, $\{T_m\}$ has a subsequence which converges uniformly in every compact subset of $\bar{R}^n - \{y\}$ for some point y possibly $y = x$. $y \in L$ according to Theorem 6 (c), and in this case the proof is completed.

If for some point $b \in \bar{R}^n$ $\{T_m(b)\}$ does not converge to x, then $\{T_m\}$ has a subsequence which we denote again by $\{T_m\}$, such that $T_m(b) \to y$, for some y, $y \neq x$. According to Theorem 3, $\{T_m\}$ is normal in $\bar{R}^n - \{x\} - \{b\}$, and it consequently has subsequence, which we denote again by $\{T_m\}$ which converges to a mapping T uniformly in any compact subset of $\bar{R}^n - \{x\} - \{b\}$. Since G is discrete, then by Theorem 4 either $T[\bar{R}^n - \{x\} - \{b\}] = x$ or $T[\bar{R}^n - \{x\} - \{b\}] = y$. In the first case $T_m(z) \to x$ uniformly in any compact subset of $\bar{R}^n - \{b\} \cdot b \in L$, because $T_m^{-1}(z) \to b$ for any $z \neq x$, according to Theorem 6(b). The proof of (ii) is then completed in denoting b by y. If the latter case holds, i.e. $T[\bar{R}^n - \{x\} - \{b\}] = y$, then, in view of Theorem 6(b), $T_m^{-1}(z) \to x$ uni-

* For otherwise T^{-1} would exist, $T_m^{-1} \to T^{-1}$ and $T_{m+1}^{-1} T_m \to I$. Note that $\{T_{m+1}^{-1} T_m\}$ has infinitely many distinct elements; otherwise $T_{m+1}^{-1} T_m = I$ for every m sufficiently large.

formly in every compact subset of $\bar{R}^n - \{y\}$. Evidently $y \in L$ and the proof is completed.

REMARK. With the aid of the last theorem one can show easily that if G is discrete and $L(G)$ has more than two points, then $L(G)$ is infinite and perfect.

REFERENCES

1. G. D. Anderson, *Symmetrization and extremal rings in space*, Ann. Acad. Sci. Fenn. **438** (1969), 1–24.

2. P. Caraman, *On the equivalence of the definitions of the n-dimensional quasiconformal homeomorphisms*, Rev. Roumaine Math. Pures Appl. XII (1967), 889-943.

3. F. W. Gehring, *Symmetrization of rings in space*, Trans. Amer. Math. Soc. **101** (1961), 499–519.

4. F. W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393.

5. F. W. Gehring, *The Caratheodory convergence theorem for quasiconformal mappings in space*, Ann. Acad. Sc. Fenn. Ser. A I, **336/11** (1963), 1-21.

6. O. Lehto and K. I. Virtanen, *Quasikonforme Abbidungen*, Springer-Verlag, 1965.

7. C. Loewner, *On the conformal capacity in space*, J. Math. Mech. **8** (1959), 411–414.

8. J. VÄISÄLÄ *Lectures on n-dimensional quasiconformal mappings*, Van Nostrand (to appear).

9. W. P. Ziemer, *Extremal length and conformal capacity*. Trans. Amer. Math. Soc., **126** (1967), 460–473.

TECHNION—ISRAEL INSTITUTE OF TECHNOLOGY
HAIFA